MATH 2040 Lecture 16 (3/11/2016)

§ Normal and self-adjoint operators

Last time: $\exists\,!$ of T^*.

Recall: $T : V \to V$ linear on $\overset{n}{\overset{\shortparallel}{\dim}}\, V < +\infty$

Q: T diagonalizable $\iff \exists$ eigenbasis β

Sufficiency tests:

I) $\exists$ n distinct e-values $\Rightarrow$ diagonalizable

II) $A = [T]_\beta$ symmetric (real) $\Rightarrow$ diagonalizable $/\mathbb{R}$.

Nec. & Suff. test:

T diagonalizable (over $\mathbb{F}$) $\iff$ ① $f(t)$ splits $/\mathbb{F}$ ② $\dim E_{\lambda_i} = m_i \quad \forall i$.

Assume: $(V, \langle , \rangle)$ inner product space $(\dim V < +\infty)$

$$T : V \to V$$

Q: Can I diagonalize T by an orthonormal eigenbasis ?

Simple answer: Spectrum Theorems

Let $T: V \to V$ be a linear op. on an inner prod. space $(V, \langle , \rangle)$ $\dim V < +\infty$.

$\exists$ orthonormal eigenbasis for T $\iff \begin{cases} T = T^* \text{ (self-adjoint) if } \mathbb{F} = \mathbb{R} \\ TT^* = T^*T \text{ (normal) if } \mathbb{F} = \mathbb{C} \end{cases}$

<u>Observe</u>: $\mathbb{F}=\mathbb{R}$, $[T]_\beta = [T^*]_\beta = [T]_\beta^t \Rightarrow$ 2nd suff. test!

β O.N.B $\quad \underbrace{\qquad\qquad}_{\text{Symmetric matrix}}$

<u>Def^n</u>: $A \in M_{n\times n}(\mathbb{F})$, $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$

$$\begin{cases} A = A^* & \text{self-adjoint} \\ AA^* = A^*A & \text{normal} \end{cases}$$

<u>Prop</u>: self-adjoint $\Rightarrow$ normal

<u>Proof</u>: $T = T^* \Rightarrow TT^* = TT = T^2 = T^*T$. $\quad \square$

<u>Note</u>: normal $\not\Rightarrow$ self-adjoint

<u>Counterexamples</u>:

1) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotate by θ (counterclockwise)

Take β = std O.N.B in $\mathbb{R}^2$

$$A = [T]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$

$$A^t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \neq A$$

so, $\underset{T}{A}$ is <u>NOT</u> self-adjoint

$$A^tA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = AA^t \Rightarrow \underset{T}{A} \text{ is normal.}$$

2) $A \in M_{n\times n}(\mathbb{R})$ skew-symmetric i.e. $A^t = -A$ $\Rightarrow$ A is normal but **NOT** self-adjoint (unless $A = 0$)

Reason: $A^tA = (-A)A = -A^2 = AA^t$ $\blacksquare$

Theorem: Let $T: V \to V$ linear on an inner prod. space $(V, \langle , \rangle)$.

If T is normal, then

(1) $\|Tx\| = \|T^*x\| \quad \forall x \in V$

(but not nec. $= \|x\|$)

(2) $T - cI$ is normal $\forall c \in \mathbb{F}$

* (3) $Tx = \lambda x \Rightarrow T^*x = \bar{\lambda} x$

* (4) $\left.\begin{array}{l} x_1 \in E_{\lambda_1}(T) \\ x_2 \in E_{\lambda_2}(T) \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \Rightarrow \langle x_1, x_2 \rangle = 0.$

If T is self-adjoint, then (1) - (4) holds and

* (5) all eigenvalues of T are $\mathbb{R}$.

Proof: $TT^* = T^*T$ normal

(1) $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle = \langle x, TT^*x \rangle$

$$= \langle T^* x, T^* x \rangle = \| T^* x \|^2 .$$

(2) $(T - cI)^*(T - cI) = (T^* - \bar{c}I)(T - cI) = T^*T - \bar{c}T - cT^* + |c|^2 I$

$$\| \checkmark$$

$(T - cI)(T - cI)^* = (T - cI)(T^* - \bar{c}I) = TT^* - \bar{c}T - cT^* + |c|^2 I$

(3) $Tx = \lambda x \Rightarrow \| Tx - \lambda x \| = 0$

$(\Leftrightarrow T^* x = \bar{\lambda} x) \Rightarrow \| (T - \lambda I) x \| = 0$

$$\overset{(1)+(2)}{\Rightarrow} \| (T - \lambda I)^* x \| = 0$$

$$\Rightarrow T^* x = \bar{\lambda} x .$$